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23

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Peter E. Kloeden Eckhard Platen

Numerical Solution of Stochastic Differential Equations

With 85 Figures

Springer

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Dedicated to Our Parents

Authors' Comments on the Revised and Updated Printing

Due to the strong interest in our book and its wide use in many different areas of application, in particular in finance, we have decided after a Second Printing to revise and update several parts of the book to be incorporated in its Third Printing. We have updated the Brief Survey of Stochastic Numerical Methods at the beginning of the book to make it even more useful to those readers who would like to get a first and up to date impression about the area. Within the main body of the book only a few misprints have had to be corrected. We would like to thank all those who pointed these out to us. The Bibliographical Notes at the end of the book have also been considerably extended. We have made an attempt to include almost all references that we know of and that appear to be relevant for the topics presented in the book. Accordingly, the list of references has been significantly increased. This also reflects the rapidly increasing literature on the topic and the growing importance of stochastic numerical methods in various fields of application.
We hope that the Third Printing of our book will continue to be a resource for teaching, research and applications. Also in future we would appreciate receiving any suggestions for further improvements.

April 1999 *Peter E. Kloeden*
Eckhard Platen

Authors' Comments on the Corrected Second Printing

The timely appearance of our book in July 1992 and its enthusiastic reception has lead to its being sold out in little more than two years. Springer's decision to reprint the book has provided us with the opportunity to correct some minor mathematical and typographical errors in the first printing of the book, as well as to update the status of papers previously listed in the References as to appear. We thank all of those readers who have kindly pointed out misprints and errors to us and would appreciate receiving any suggestions for further improvements that could be incorporated into a future revised edition of the book.

March 1995

Preface

The aim of this book is to provide an accessible introduction to stochastic differential equations and their applications together with a systematic presentation of methods available for their numerical solution.

During the past decade there has been an accelerating interest in the development of numerical methods for stochastic differential equations (SDEs). This activity has been as strong in the engineering and physical sciences as it has in mathematics, resulting inevitably in some duplication of effort due to an unfamiliarity with the developments in other disciplines. Much of the reported work has been motivated by the need to solve particular types of problems, for which, even more so than in the deterministic context, specific methods are required. The treatment has often been heuristic and ad hoc in character. Nevertheless, there are underlying principles present in many of the papers, an understanding of which will enable one to develop or apply appropriate numerical schemes for particular problems or classes of problems.

The present book does not claim to be a complete or an up to date account of the state of the art of the subject. Rather, it attempts to provide a systematic framework for an understanding of the basic concepts and of the basic tools needed for the development and implementation of numerical methods for SDEs, primarily time discretization methods for initial value problems of SDEs with Ito diffusions as their solutions. In doing so we have selected special topics and many recent results to illustrate these ideas, to help readers see potential developments and to stimulate their interest to contribute to the subject from the perspective of their own discipline and its particular requirements. The book is thus directed at readers from quite different fields and backgrounds. We envisage three broad groups of readers who may benefit from the book:

(i) those just interested in modelling and applying standard methods, typically from the social and life sciences and often without a strong background in mathematics;

(ii) those with a technical background in mathematical methods typical of engineers and physicists who are interested in developing new schemes as well as implementing them;

(iii) those with a stronger, advanced mathematical background, such as stochasticians, who are more interested in theoretical developments and underlying mathematical issues.

The book is written at a level that is appropriate for a reader with an engineer's or physicist's undergraduate training in mathematical methods. Many chapters begin with a descriptive overview of their contents which may be accessible to those from the first group of readers mentioned above. There are also several more theoretical sections and chapters for the more mathematically inclined reader. In the "Suggestions for the Reader" we provide some hints for each of the three groups of readers on how to use the different parts of the book.

We have tried to make the exposition as accessible to as wide a readership as possible. The first third of the book introduces the reader to the theory of stochastic differential equations with minimal use of measure theoretic concepts. The reader will also find an extensive list of explicit solutions for SDEs. The application of SDEs in important fields such as physics, engineering, biology, communications, economics, finance, ecology, hydrology, filtering, control, genetics, etc, is emphasized and examples of models involving SDEs are presented. In addition, the use of the numerical methods introduced in the book is illustrated for typical problems in two separate chapters.

The book consists of 17 Chapters, which are grouped into 6 Parts. Part I on Preliminaries provides background material on probability, stochastic processes and statistics. Part II on Stochastic Differential Equations introduces stochastic calculus, stochastic differential equations and stochastic Taylor expansions. These stochastic Taylor expansions provide a universally applicable tool for SDEs which is analogous to the deterministic Taylor formula in ordinary calculus. Part III on Applications of Stochastic Differential Equations surveys the application of SDEs in a diversity of disciplines and indicates the essential ideas of control, filtering, stability and parametric estimation for SDEs. The investigation of numerical methods begins in Part IV on Time Discrete Approximations with a brief review of time discretization methods for ordinary differential equations and an introduction to such methods for SDEs. For the latter we use the simple Euler scheme to highlight the basic issues and types of problems and objectives that arise when SDEs are solved numerically. In particular, we distinguish between strong and weak approximations, depending on whether good pathwise or good probability distributional approximations are sought. In the remaining two parts of the book different classes of numerical schemes appropriate for these tasks are developed and investigated. Stochastic Taylor expansions play a central role in this development. Part V is on Strong Approximations and Part VI on Weak Approximations. It is in these two Parts that the schemes are derived, their convergence orders and stability established, and various applications of the schemes considered.

Exercises are provided in most sections to nurture the reader's understanding of the material under discussion. Solutions of the Exercises can be found at the end of the book.

Many PC-Exercises are included throughout the book to assist the reader to develop "hands on" numerical skills and an intuitive understanding of the basic concepts and of the properties and the issues concerning the implementation of the numerical schemes introduced. These PC-Exercises often build on earlier ones and reappear later in the text and applications, so the reader is encouraged to work through them systematically. The companion book

P. E. Kloeden, E. Platen and H. Schurz: *The Numerical Solution of Stochastic Differential Equations through Computer Experiments.* Springer (1993).

contains programs on a floppy disc for these PC–Exercises and a more detailed discussion on their implementation and results. Extensive simulation studies can also be found in this book.

To simplify the presentation we have concentrated on Ito diffusion processes and have intentionally not considered some important advanced concepts and results from stochastic analysis such as semimartingales with jumps or boundaries or SDEs on manifolds. For a more theoretical and complete treatment of stochastic differential equations than we give here we refer readers to the monograph

N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes.* North-Holland, Amsterdam (1981; 2nd Edition, 1989).

In the few instances that we shall require advanced results in a proof we shall state a reference explicitly in the text. In addition, in the case studies of different applications of SDEs and numerical methods in Chapters 7, 13 and 17 we shall indicate the names of the authors of the papers that we have consulted. Otherwise, and in general, further information and appropriate references for the section under consideration will be provided in the Bibliographical Remarks at the end of the book.

Two types of numbering system are used throughout the book. Equations are numbered by their section and number in the section, for example (2.1), and are referred to as such in this section and within the chapter which includes it; the chapter number appears as a prefix when the equation is referred to in other chapters. The resulting numbers, (2.1) or (3.2.1) say, will always appear in parentheses. Examples, Exercises, PC–Exercises, Remarks, Theorems and Corollaries are all numbered by their chapter, section and order of occurrence regardless of qualifier. They will always be prefixed by their qualifier and never appear in parentheses, for example Theorem 3.2.1. Figures and Tables are each, and separately, numbered by the same three number system, with the third number now referring only to the occurrence of the Figure or the Table, respectively. The only exception to these numbering systems is in the "Brief Survey of Stochastic Numerical Methods" at the beginning of the book, where just a single number is used for each equation.

During the writing of this book we have received much encouragement, support and constructive criticism from a large number of sources. In particular, we mention with gratitude L. Arnold, H. Föllmer, J. Gärtner, C. Heyde, G. Kallianpur, A. Pakes, M. Sørenson and D. Talay, as well as each others' institutions, the Institute for Dynamical Systems at the University of Bremen, the Institute of Advanced Studies at the Australian National University and the Institute for Applied Mathematics of the University of Hamburg. Special thanks also go to H. Schurz and N. Hofmann who programmed and tested the PC-Exercises in the book and produced the figures.

Berlin, May 1991

Peter E. Kloeden
Eckhard Platen

Contents

Suggestions for the Reader

We mentioned in the Preface that we have tried to arrange the material of this book in a way that would make it accessible to as wide a readership as possible. Since prospective readers will undoubtedly have different backgrounds and objectives, the following hints may facilitate their use of the book.

(i) We begin with those readers who require only sufficient understanding of stochastic differential equations to be able to apply them and appropriate numerical methods in different fields of application. The deeper mathematical issues are avoided in the following flowchart which provides a reading guide to the book for those without a strong background in mathematics.

§1.1 → §1.2 → §1.3 → §1.6 → §1.7 → §1.8

↓

§3.1

↓

§4.1 → §4.4

↓

Chapter 6

↓

Chapter 7

↓

Chapter 8

↓

§9.1 → §9.2 → §9.3 → §9.4 → §9.6 → §9.7

↓

§10.1 → §10.2 → §10.3 → §10.4 → §10.5

↓

§11.1 → §11.2 → §11.3 → §11.4

↓

§12.1 → §12.2 → §12.3 → §12.4

↓

§13.1 → §13.2 → §13.3

↓

§14.1 → §14.2 → §14.3 → §14.4

↓

§15.1 → §15.2 → §15.3 → §15.4 → §15.5

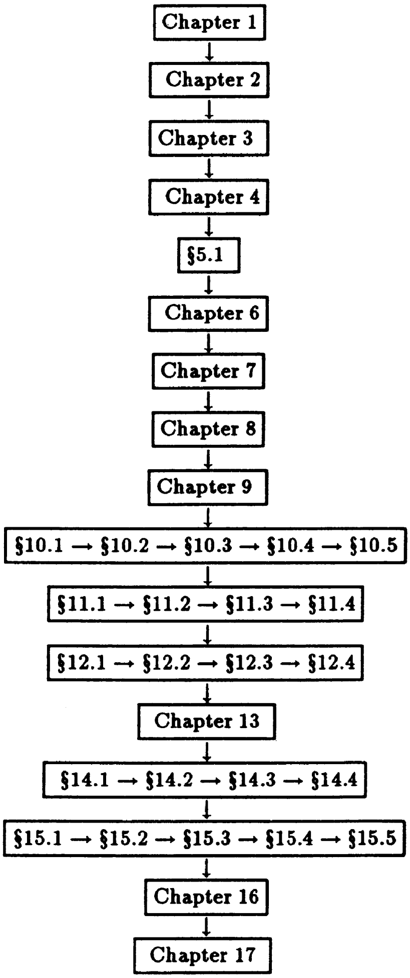
↓

§16.1

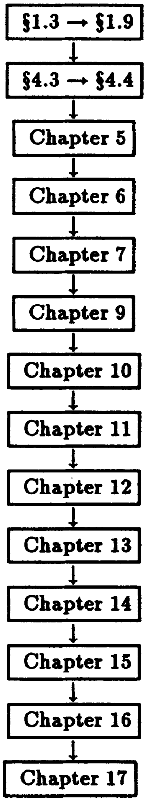
↓

Chapter 17

(ii) Engineers, physicists and others with a more technical background in mathematical methods who are interested in applying stochastic differential equations and in implementing efficient numerical schemes or developing new schemes for specific classes of applications, could use the book according to the following flowchart. This now includes more material on the underlying mathematical techniques without too much emphasis on proofs.



(iii) Mathematicians and other readers with a stronger mathematical background may omit the introductory parts of the book. The following flowchart emphasizes the deeper, more theoretical aspects of the numerical approximation of Ito diffusion processes while avoiding well known or standard topics.



Basic Notation

$\emptyset$	the empty set
$a \in A$	a is an element of the set A
$a \notin A$	a is not an element of the set A
A^c	the complement of the set A
$A \cup B$	the union of sets A and B
$A \cap B$	the intersection of sets A and B
$A \setminus B$	the set of elements of set A that are not in set B
$:=$	defined as or denoted by
$\equiv$	identically equal to
$\approx$	approximately equal to
$\sim$	with distribution
$\Re$	the set of real numbers
$\Re^+$	the set of non-negative real numbers
(a, b)	the open interval $a < x < b$ in $\Re$
$[a, b]$	the closed interval $a \leq x \leq b$ in $\Re$
$a \vee b$	the maximum of a and b
$a \wedge b$	the minimum of a and b
$n!$	the factorial of the positive integer n
$[a]$	the largest integer not exceeding a
$\Re^d$	the d-dimensional Euclidean space
$x = (x^1, \ldots, x^d)$	a vector $x \in \Re^d$ with ith component x^i for $i = 1, \ldots, d$
(x, y)	the scalar product of vectors x, $y \in \Re^d$
$\lvert x \rvert$	the Euclidean norm of a vector $x \in \Re^d$
$x^\top$	transpose of the vector x
$A = [a^{i,j}]$	a matrix A with ijth component $a^{i,j}$
$\imath$	the square root of -1
$\mathrm{Re}(z)$	the real part of a complex number z
$\mathrm{Im}(z)$	the imaginary part of a complex number z

$f : Q_1 \to Q_2$	a function f from Q_1 into Q_2
1_A	the indicator function of the set A
f'	the first derivative of a function $f : \Re^1 \to \Re^1$
f''	the second derivative of a function $f : \Re^1 \to \Re^1$
$f^{(k)}$	the kth derivative of a function $f : \Re^1 \to \Re^1$
$\partial_{x^i} u$, $\frac{\partial u}{\partial x^i}$	the ith partial derivative of a function $u : \Re^d \to \Re^1$
$\partial^k_{x^i} u$, $\left(\frac{\partial}{\partial x^i}\right)^k u$	the kth order partial derivative of u with respect to x^i
$\mathcal{C}(\Re^m, \Re^n)$	the space of continuous functions $f : \Re^m \to \Re^n$
$\mathcal{C}^k(\Re^m, \Re^n)$	the space of k times continuously differentiable functions $f : \Re^n \to \Re^n$
$\mathcal{B}$	the σ-algebra of Borel subsets of $\Re^1$
$\mathcal{L}$	the σ-algebra of Lebesgue subsets of $\Re^1$
$E(X)$	the expectation of the random variable X
$\delta_{i,j}$	the Kronecker delta symbol
$O(r^p)$	expression divided by r^p remains bounded as $r \to 0$
$o(r^p)$	expression divided by r^p converges to zero as $r \to 0$
a.s.	almost surely
w.p.1	with probability 1

Other notation will be defined where it is first used. Note that vectors and matrices will usually be indexed with superscripts. Parentheses will then be used when taking powers of their components, for example with $(x^i)^3$ denoting the cube of x^i. Square brackets $[\cdot]$ will often be used to visually simplify nested expressions, with the few instances where it denotes the integer part of a real number being indicated in the text. Function space norms will always be written with double bars $\|\cdot\|$, often with a distinguishing subscript.

Brief Survey of Stochastic Numerical Methods

An Ito process $X = \{X_t, t \geq 0\}$ has the form

$$X_t = X_0 + \int_0^t a(X_s)\, ds + \int_0^t b(X_s)\, dW_s \tag{1}$$

for $t \geq 0$. It consists of an initial value $X_0 = x_0$, which may be random, a slowly varying continuous component called the drift and a rapidly varying continuous random component called the diffusion. The second integral in (1) is an Ito stochastic integral with respect to the Wiener process $W = \{W_t,\, t \geq 0\}$. The integral equation (1) is often written in the differential form

$$dX_t = a(X_t)\, dt + b(X_t)\, dW_t \tag{2}$$

and is then called an Ito stochastic differential equation (SDE). For simplicity, in this survey we shall restrict our attention to a 1-dimensional Ito process X with a 1-dimensional driving Wiener process W.

Unfortunately explicitly solvable SDEs such as those listed in Section 4 of Chapter 4 are rare in practical applications. There are, however, now an increasing number of numerical methods for the solution of SDEs mentioned in the literature. A crucial task is the systematic development of efficient numerical methods for SDEs, a task to which this book is addressed. Obviously such methods should be implementable on digital computers. They often involve the simulation of a large number of different sample paths in order to estimate various statistical features of the desired solution. Modern supercomputers with their parallel architecture are well suited to such calculations; see Petersen (1987) and Hausenblas (1999b).

Here we shall survey various time discrete numerical methods which are appropriate for the simulation of sample paths or functionals of Ito processes.

Numerical Approaches to Stochastic Differential Equations

To begin we shall briefly mention several different approaches that have been suggested for the numerical solution of SDEs. On the very general level there is a method due to Boyce (1978) by means of which one can investigate, in principle at least, general random systems by Monte Carlo methods. For SDEs this method is somewhat inefficient because it does not use the special structure of these equations, specifically their characterization by their drift and diffusion coefficients.

Kushner (1974) and Kushner & Dupuis (1992) proposed the discretization of both time and space variables, so the approximating processes are then finite state Markov chains. These can be handled on digital computers through their transition matrices. Higher order Markov chain approximations are developed in Platen (1992). In comparison with the information encompassed succinctly in the drift and diffusion coefficients of an SDE, transition matrices contain a considerable amount of superfluous information which must be repeatedly reprocessed during computations. Consequently such a Markov chain approach seems applicable only for low dimensional problems on bounded domains. Similar disadvantages also arise, in higher dimensions at least, when standard numerical methods are used to solve parabolic partial differential equations, such as the Fokker-Planck equation and its adjoint, associated with functionals of the solutions of SDEs. These are, of course, also methods for computing the probability densities of Ito diffusions.

The most efficient and widely applicable approach to solving SDEs seems to be the simulation of sample paths of time discrete approximations on digital computers. This is based on a finite discretization of the time interval $[0, T]$ under consideration and generates approximate values of the sample paths step by step at the discretization times. The simulated sample paths can then be analysed by usual statistical methods to determine how good the approximation is and in what sense it is close to the exact solution. The state variables here are not discretized as in Kushner's Markov chain approach and the structure of the SDE as provided by the drift and diffusion coefficients is used in a natural way. An advantage of considerable practical importance of this approach is that the computational costs such as time and memory required increase only polynomially with the dimension of the problem. A multi-faceted variety of reseach topics on numerical methods for SDEs has emerged over the last twenty years. Many of these can be linked to complexity theory, see e.g., Traub, Wasilkowski & Wozniakowski (1988), Wozniakowski (1991) and Sloan & Wozniakowski (1998), where it was shown that simulation approaches, including those of stochastic numerical analysis, are optimal with respect to average case complexity.

Time Discrete Approximations

Simulation studies and theoretical investigations by Clements & Anderson (1973), Wright (1974), Fahrmeir (1976), Clark & Cameron (1980), Rümelin (1982) and others showed that not all heuristic time discrete approximations of an SDE converge in a useful sense to the solution process as the maximum step size δ tends to zero. In particular, it was found that one cannot simply use a deterministic numerical method for ordinary differential equations, such as a higher order Runge-Kutta method. Consequently a careful and systematic investigation of different methods is needed in order to select a sufficiently efficient method for the task at hand.

We shall consider a time discretization $(\tau)_\delta$ with

(3) $$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$$

of a time interval $[0, T]$, which in the simplest equidistant case has step size

(4) $$\delta = \frac{T}{N}.$$

We shall see in Chapter 9 that general time discretizations, even with random times, are possible, but usually a maximum step size δ must be specified.
The simplest heuristic time discrete approximation is the stochastic generalization of the Euler approximation which is sometimes called the *Euler-Maruyama approximation*, see Maruyama (1955), but often just the *Euler approximation*. For the SDE (2) it has the form

(5) $$Y_{n+1} = Y_n + a\,(Y_n)\,\Delta_n + b\,(Y_n)\,\Delta W_n$$

for $n = 0, 1, \ldots, N-1$ with initial value

(6) $$Y_0 = x_0,$$

where

(7) $$\Delta_n = \tau_{n+1} - \tau_n = \delta$$

and

(8) $$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$$

for $n = 0, 1, \ldots, N-1$. Essentially, it is formed by fixing the integrands in (1) to their values at the beginning of each discretization time subinterval.
The recursive scheme (5) obviously gives values of the approximation only at the discretization times. If values are required at intermediate instants, then either piecewise constant values from the preceding discretization point or some interpolation, especially a linear interpolation, of the values of the two immediate enclosing discretization points could be used.
The random variables ΔW_n defined in (8) are independent $N(0; \Delta_n)$ normally distributed random variables, that is with means and variances

$$E\,(\Delta W_n) = 0 \qquad \text{and} \qquad E\left((\Delta W_n)^2\right) = \Delta_n,$$

respectively, for $n = 0, 1, \ldots, N-1$. In simulations we can generate such random variables from independent, uniformly distributed random variables on $[0, 1]$, which are usually provided by a pseudo-random number generator on a digital computer. We shall discuss and test random number generators in Sections 3 and 9 of Chapter 1.
In practice, linear or non-linear congruential pseudo-random number generators are often used. An introduction to this area is given by Ripley (1983a). Books that include chapters on random number generation include Ermakov

(1975), Yakowitz (1977), Rubinstein (1981), Ripley (1983b), Morgan (1984), Ross (1990), Mikhailov (1992), Fishman (1996) and Gentle (1998). We mention also the papers by Box & Muller (1958), Marsaglia & Bray (1964), Brent (1974), Eichenauer & Lehn (1986), Niederreiter (1988), Sugita (1995), Antipov (1995) and Antipov (1996).

The Strong Convergence Criterion

In problems such as those that we shall consider in Chapter 13 involving direct simulations, filtering or testing estimators of Ito processes it is important that the trajectories, that is the sample paths, of the approximation be close to those of the Ito process. This suggests that a criterion involving some form of strong convergence should be used. Mathematically it is advantageous to consider the absolute error at the final time instant T, that is

$$\epsilon(\delta) = E\left(|X_T - Y_N|\right), \tag{9}$$

which can be estimated from the root mean square error via the Lyapunov inequality

$$\epsilon(\delta) = E\left(|X_T - Y_N|\right) \leq \sqrt{E\left(|X_T - Y_N|^2\right)}. \tag{10}$$

The absolute error (9) is certainly a criterion for the closeness of the sample paths of the Ito process X and the approximation Y at time T.
We shall say that an approximating process Y *converges in the strong sense with order* $\gamma \in (0, \infty]$ if there exists a finite constant K and a positive constant δ_0 such that

$$E\left(|X_T - Y_N|\right) \leq K\,\delta^{\gamma} \tag{11}$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$. In the deterministic case with zero diffusion coefficient $b \equiv 0$ this strong convergence criterion reduces to the usual deterministic criterion for the approximation of ordinary differential equations. The order of a scheme is sometimes less in the stochastic case than in the corresponding deterministic one, essentially because the increments ΔW_n of the Wiener process are of root mean square order $\delta^{1/2}$ and not δ. In fact, the Euler approximation (5) for SDEs has strong order $\gamma = 0.5$ in contrast with the order 1.0 of the Euler approximation for ordinary differential equations.
Publications related to the development of higher order strong approximations include Franklin (1965), Shinozuka (1971), Kohler & Boyce (1974), Rao, Borwanker & Ramkrishna (1974), Dsagnidse & Tschitashvili (1975), Harris (1976), Glorennec (1977), Kloeden & Pearson (1977), Clark (1978), Nikitin & Razevig (1978), Helfand (1979), Platen (1980a), Razevig (1980), Greenside & Helfand (1981), Casasus (1982), Clark (1982), Guo (1982), Talay (1982a, 1982b, 1982c, 1983), Drummond, Duane & Horgan (1983), Casasus (1984), Guo (1984), Janssen (1984a, 1984b), Shimizu & Kawachi (1984), Tetzlaff & Zschiesche

(1984), Unny (1984), Clark (1985), Averina & Artemiev (1986, 1988), Drummond, Hoch & Horgan (1986), Kozlov & Petryakov (1986), Greiner, Strittmatter & Honerkamp (1987), Liske & Platen (1987), Platen (1987), Milstein (1987, 1988b), Shkurko (1987), Römisch & Wakolbinger (1987), Golec & Ladde (1989), Feng (1990), Nakazawa (1990), Bensoussan, Glowinski & Rascanu (1992), Feng, Lei & Qian (1992), Artemiev (1993b), Kloeden, Platen & Schurz (1993), Saito & Mitsui (1993a, 1996), Petersen (1994b), Török (1994), Ogawa (1995), Gelbrich & Rachev (1996), Grecksch & Wadewitz (1996), Newton (1996), Schurz (1996b), Yannios & Kloeden (1996), Artemiev & Averina (1997), Denk & Schäffer (1997), Abukhaled & Allen (1998) and Schein & Denk (1998).

The Weak Convergence Criterion

In many practical situations, some of which will be described in Chapter 17, it is not necessary to have a close pathwise approximation of an Ito process. Often one may only be interested in some function of the value of the Ito process at a given final time T such as one of the first two moments $E(X_T)$ and $E((X_T)^2)$ or, more generally, the expectation $E(g(X_T))$ for some function g. In simulating such a functional it suffices to have a good approximation of the probability distribution of the random variable X_T rather than a close approximation of sample paths. Thus the type of approximation required here is much weaker than that provided by the strong convergence criterion.

We shall say that a time discrete approximation Y *converges in the weak sense with order* $\beta \in (0, \infty]$ if for any polynomial g there exists a finite constant K and a positive constant δ_0 such that

$$|E(g(X_T)) - E(g(Y_N))| \leq K\, \delta^{\beta} \tag{12}$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$. In Section 7 of Chapter 9 we shall generalize slightly the class of test fuctions g used here. When the diffusion coefficient in (1) vanishes, this weak convergence criterion with $g(x) \equiv x$ also reduces to the usual deterministic convergence criterion for ordinary differential equations.

Under assumptions of sufficient regularity Milstein (1978) showed that an Euler approximation of an Ito process converges with weak order $\beta = 1.0$, which is greater than its strong order of convergence $\gamma = 0.5$. On the other hand, Mikulevicius & Platen (1991) proved that the Euler scheme still converges, but with weak order less than 1.0, when the coefficients of (1) are only Hölder continuous, that is Lipschitz-like with a fractional power. Some of the papers in which the Euler method has been studied include Allain (1974), Yamada (1976), Gikhman & Skorokhod (1979), Clark & Cameron (1980), Ikeda & Watanabe (1989), Janssen (1984a, 1984b), Atalla (1986), Jacod & Shiryaev (1987), Kaneko & Nakao (1988), Kanagawa (1988, 1989, 1995, 1996, 1997), Golec & Ladde (1989), Mackevicius (1994), Cambanis & Hu (1996), Gelbrich

(1995), Bally & Talay (1995, 1996a, 1996b), Jacod & Protter (1998), Kohatsu-Higa & Ogawa (1997) and Chan & Stramer (1998).
Higher order weak approximations have been investigated, e.g., by Milstein (1978), Platen (1984), Talay (1984), Mikulevicius & Platen (1991). In particular, weak approximations of the Runge-Kutta type have been proposed and studied by Greenside & Helfand (1981), Talay (1984), Platen (1984), Klauder & Petersen (1985a), Milstein (1985), Haworth & Pope (1986), Averina & Artemiev (1986), Mackevicius (1994) and Komori & Mitsui (1995). Wagner (1987b) has investigated the use of unbiased weak approximations, that is with $\beta = \infty$, for estimating functionals of Ito diffusions.

Stochastic Taylor Formulae

A natural way of classifying numerical methods for SDEs is to compare them with strong and weak Taylor approximations. The increments of such approximations are obtained by truncating the stochastic Taylor formula, also called Wagner-Platen formula, see Wagner & Platen (1978). This result was then extended and generalised in Platen (1981a, 1982b), Platen & Wagner (1982), Azencott (1982), Sussmann (1988), Yen (1988, 1992), BenArous (1989), Kloeden & Platen (1991a, 1991b), Hu (1992, 1996), Hu & Watanabe (1996), Kohatsu-Higa (1997), Liu & Li (1997) and Kuznetsov (1998).
A Stratonovich version of the stochastic Taylor formula was presented in Kloeden & Platen (1991a, 1991b) and can be found together with results on multiple stochastic integrals in Chapter 5.
The Wagner-Platen formula allows a function of an Ito process, that is $f(X_t)$, to be expanded about $f(X_{t_0})$ in terms of multiple stochastic integrals weighted by coefficients which are evaluated at X_{t_0}. These coefficients are formed from the drift and diffusion coefficients of the Ito process and their derivatives up to some specified order. The remainder term in the formula contains a finite number of multiple stochastic integrals of the next higher multiplicity, but now with nonconstant integrands. For example, a Wagner-Platen formula for the Ito process (1) for $t \in [t_0, T]$ may have the form

$$
\begin{aligned}
(13) \qquad f(X_t) &= f(X_{t_0}) + c_1(X_{t_0}) \int_{t_0}^{t} ds + c_2(X_{t_0}) \int_{t_0}^{t} dW_s \\
&\quad + c_3(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s_2} dW_{s_1}\, dW_{s_2} + R
\end{aligned}
$$

with coefficients

$$
\begin{aligned}
c_1(x) &= a(x) f'(x) + \frac{1}{2} (b(x))^2 f''(x), \\
c_2(x) &= b(x) f'(x), \\
c_3(x) &= b(x) \{ b(x) f''(x) + b'(x) f'(x) \}.
\end{aligned}
$$

Here the remainder R consists of higher order multiple stochastic integrals with nonconstant integrands involving the function f, the drift and diffusion coefficients and their derivatives. A Wagner-Platen formula can be thought of as a generalization of both the Ito formula and the deterministic Taylor formula. If we use the function $f(x) \equiv x$ in the formula (13) we obtain the following representation for the Ito process (1):

$$(14) \qquad X_t = X_{t_0} + a\left(X_{t_0}\right) \int_{t_0}^{t} ds + b\left(X_{t_0}\right) \int_{t_0}^{t} dW_s + b\left(X_{t_0}\right) b'\left(X_{t_0}\right) \int_{t_0}^{t} \int_{t_0}^{s_2} dW_{s_1}\, dW_{s_2} + R.$$

By truncating stochastic Taylor expansions such as (14) about successive discretization points we can form time discrete Taylor approximations which we may interpret as basic numerical schemes for an SDE. In addition we can compare other schemes, such as those of the Runge-Kutta type, with time discrete Taylor approximations to determine their order of strong or weak convergence. We shall see that we must include the appropriate terms from the corresponding stochastic Taylor expansion, that is the necessary higher multiple stochastic integrals, to obtain a numerical scheme with a higher order of strong or weak convergence. Thus to build a higher order scheme one does not only need more smoothness of the drift and diffusion coefficients but also more information about the driving Wiener processes.

Strong Taylor Approximations

The simplest strong Taylor approximation of an Ito diffusion is the *Euler approximation*

$$(15) \qquad Y_{n+1} = Y_n + a\,\Delta_n + b\,\Delta W_n$$

for $n = 0, 1, \ldots, N-1$ with initial condition (6), where Δ_n and ΔW_n are defined by (7) and (8), respectively, with the ΔW_n representing independent $N(0;\Delta_n)$ normally distributed random variables. Here we have written a for $a(Y_n)$ and b for $b(Y_n)$, a convention which we shall henceforth use for any function. In addition, as here, we shall not repeat the standard initial condition (6) in what follows. It was shown in Gikhman & Skorokhod (1972a) that the Euler scheme converges with strong order $\gamma = 0.5$ under Lipschitz and bounded growth conditions on the coefficients a and b.

If we include the next term from the Wagner-Platen formula (14) in the scheme (15) we obtain the *Milstein scheme*

$$(16) \qquad Y_{n+1} = Y_n + a\,\Delta_n + b\,\Delta W_n + \frac{1}{2} b\,b' \left\{ (\Delta W_n)^2 - \Delta_n \right\}$$

for $n = 0, 1, \ldots, N-1$; see Milstein (1974). The additional term here is from the double Wiener integral in (14), which can be easily computed from the

Wiener increment ΔW_n since

$$(17) \qquad \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, dW_{s_2} = \frac{1}{2}\left\{(\Delta W_n)^2 - \Delta_n\right\}.$$

We shall see that the Milstein scheme (16) converges with strong order $\gamma = 1.0$ under the assumption that $E((X_0)^2) < \infty$, that a and b are twice continuously differentiable, and that a, a', b, b' and b'' satisfy a uniform Lipschitz condition. For a multi-dimensional driving Wiener process $W = (W^1, \ldots, W^m)$ the generalization of the Milstein scheme (16) involves the multiple Wiener integrals

$$(18) \qquad I_{(j_1,j_2)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}^{j_1}\, dW_{s_2}^{j_2},$$

for $j_1, j_2 \in \{1, \ldots, m\}$ with $j_1 \neq j_2$, which cannot be expressed simply as in (16) in terms of the increments $\Delta W_n^{j_1}$ and $\Delta W_n^{j_2}$ of the corresponding Wiener processes. In Section 8 of Chapter 5 we shall suggest one possible way of approximating higher order multiple stochastic integrals like (18).

Close relationships exist between multiple Ito and Stratonovich integrals which form some kind of algebra. This algebra and certain approximations of multiple stochastic integrals have been described in Platen & Wagner (1982), Liske (1982), Platen (1984), Milstein (1988a, 1995a), Kloeden & Platen (1991a, 1991b), Kloeden, Platen & Wright (1992), Hu & Meyer (1993), Hofmann (1994), Gaines & Lyons (1994), Gaines (1994, 1995a), Castell & Gaines (1995), Li & Liu (1997), Burrage (1998) and Kuznetsov (1998).

Generally speaking we obtain more accurate strong Taylor approximations by including additional multiple stochastic integral terms from a stochastic Taylor expansion. Such integrals contain additional information about the sample paths of the Wiener process over the discretization subintervals. Their presence is a fundamental difference between the numerical analysis of stochastic and ordinary differential equations. For example, the *strong Taylor approximation of order* $\gamma = 1.5$ is given by

$$\begin{aligned}(19) \quad Y_{n+1} \;=\;& Y_n + a\,\Delta_n + b\,\Delta W_n + \frac{1}{2} b b' \left\{(\Delta W_n)^2 - \Delta_n\right\} \\ & + b a'\, \Delta Z_n + \frac{1}{2}\left\{a a' + \frac{1}{2} b^2 a''\right\} \Delta_n^2 \\ & + \left\{a b' + \frac{1}{2} b^2 b''\right\} \left\{\Delta W_n\, \Delta_n - \Delta Z_n\right\} \\ & + \frac{1}{2} b \left\{b b'' + (b')^2\right\} \left\{\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right\} \Delta W_n\end{aligned}$$

for $n = 0, 1, \ldots, N-1$. Here the additional random variable ΔZ_n is required to represent the double integral

$$(20) \qquad \Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2,$$

which is normally distributed with mean, variance and correlation

$$E(\Delta Z_n) = 0, \qquad E\left((\Delta Z_n)^2\right) = \frac{1}{3}(\Delta_n)^3 \quad \text{and} \quad E(\Delta W_n \, \Delta Z_n) = \frac{1}{2}(\Delta_n)^2,$$

respectively. All other multiple stochastic integrals appearing in the truncated Taylor expansion used to derive (19) can be expressed in terms of Δ_n, ΔW_n and ΔZ_n, thus resulting in (19). It was shown in Wagner & Platen (1978) and Platen (1981a) that the scheme (19) converges with strong order $\gamma = 1.5$ when the coefficients a and b are sufficiently smooth and satisfy Lipschitz and bounded growth conditions. We note that there is no difficulty in generating the pair of correlated normally distributed random variables ΔW_n, ΔZ_n using the transformation

$$\text{(21)} \qquad \Delta W_n = \zeta_{n,1} \, \Delta_n^{1/2} \qquad \text{and} \qquad \Delta Z_n = \frac{1}{2}\left(\zeta_{n,1} + \frac{1}{\sqrt{3}} \zeta_{n,2}\right) \Delta_n^{3/2},$$

where $\zeta_{n,1}$ and $\zeta_{n,2}$ are independent normally $N(0;1)$ distributed random variables.

Following Platen (1981a), we shall describe in Chapter 10 how schemes of any desired order of strong convergence can be constructed from the corresponding strong Taylor approximations. The implementation of such schemes requires the generation of multiple stochastic integrals such as $I_{(j_1,j_2)}$ and of higher multiplicity, which can be done by means of an approximation method which we shall describe in Chapter 5. Those readers who do not wish to use such multiple stochastic integrals could follow Clark (1978) and Newton (1986a, 1986b), in which schemes only involving the increments of the Wiener process are proposed. These schemes, which we shall describe in Section 4 of Chapter 13, are similar to the strong Taylor approximations above, but with the random variables modified. Moreover, they are optimal within the classes of strong orders $\gamma = 0.5$ or 1.0, respectively.

Strong Runge-Kutta, Two-Step and Implicit Approximations

A practical disadvantage of the above strong Taylor approximations is that the derivatives of various orders of the drift and diffusion coefficients must be determined and then evaluated at each step in addition to the coefficients themselves. There are time discrete approximations which avoid the use of derivatives, which we shall call Runge-Kutta schemes in analogy with similar schemes for ordinary differential equations. However, we emphasize that it is not always possible to use heuristic adaptations of deterministic Runge-Kutta schemes for SDEs because of the difference between ordinary and stochastic calculi.

A *strong order* 1.0 *Runge-Kutta scheme* is given by

$$\text{(22)} \quad Y_{n+1} = Y_n + a\,\Delta_n + b\,\Delta W_n + \frac{1}{2}\left\{b\left(\hat{\Upsilon}_n\right) - b\right\}\left\{(\Delta W_n)^2 - \Delta_n\right\}\Delta_n^{-1/2}$$

with supporting value

$$\hat{\Upsilon}_n = Y_n + b\,\Delta_n^{1/2}$$

for $n = 0, 1, \ldots, N-1$. This scheme can be obtained heuristically from the Milstein scheme (16) simply by replacing the derivative there by the corresponding finite difference; see Platen (1984). Clark & Cameron (1980) and Rümelin (1982) have shown that Runge-Kutta schemes like (22) converge strongly with at most order $\gamma = 1.0$. More general Runge-Kutta schemes can be found in Chapter 11, but they have usually only the strong order of convergence $\gamma = 1.0$ if just the increments ΔW_n of the Wiener process are used. Higher multiplicity stochastic integrals must be used to obtain a higher order of strong convergence.

In Rümelin (1982), Gard (1988), Kloeden & Platen (1995, 1992) and Artemiev (1993a, 1993b) further Runge-Kutta type schemes can be found. It is natural to ask whether the tree approach developed in Butcher (1987) can be translated to the stochastic setting. Some results along these lines were given by Saito & Mitsui (1993b), Burrage & Platen (1994), Komori, Saito & Mitsui (1994), Komori & Mitsui (1995), Saito & Mitsui (1996), Burrage & Burrage (1996, 1997), Burrage, Burrage & Belward (1997), Komori, Mitsui & Sugiura (1997) and Burrage (1998). For instance, in the case of a single driving Wiener process, a rooted tree methodology has been described for Stratonovich SDEs by Burrage (1998).

Four-stage Runge-Kutta methods of strong order $\gamma = 1.5$ can also be found in Burrage (1998). Similarly, in the context of filtering problems Newton (1986a, 1986b, 1991) and also Castell & Gaines (1996) have proposed approximations that are, in some sense, asymptotically efficient with respect to the leading error coefficient within a class of Runge-Kutta type methods.

Lépingle & Ribémont (1991) suggested a two-step strong scheme of first order. For the case of additive noise, $b \equiv$ *const.*, another *two-step order* 1.5 *strong scheme* which is due to the authors takes the form

$$(23) \quad Y_{n+1} = Y_n + 2a\,\Delta_n - a'\,(Y_{n-1})\,b\,(Y_{n-1})\,\Delta W_{n-1}\,\Delta_n + V_n + V_{n-1}$$

with

$$V_n = b\,\Delta W_n + a'b\,\Delta Z_n,$$

where ΔW_n and ΔZ_n are the same as in (21); see Chapter 12 for more details. What really matters in a numerical scheme is that it should be numerically stable, can be conveniently implemented, and generates fast highly accurate results. The well-known concept of *A-stability*, see Björck & Dahlquist (1974), can be directly generalised to the case of SDEs with additive noise, that is $b(x) = \text{const.}$ in equation (1), see Milstein (1988a, 1995a), Hernandez & Spigler (1992) or Kloeden & Platen (1992). A typical *implicit order* 1.5 *strong scheme* for additive noise is

$$(24) \quad Y_{n+1} = Y_n + \frac{1}{2}\,\{a\,(Y_{n+1}) + a\}\,\Delta_n + b\,\Delta W_n$$

$$+\frac{1}{2}\left\{a\left(\hat{\Upsilon}_n^+\right)-a\left(\hat{\Upsilon}_n^-\right)\right\}\left\{\Delta Z_n-\frac{1}{2}\Delta W_n\,\Delta_n\right\}\,\Delta_n^{-1/2}$$

with supporting values

$$\hat{\Upsilon}_n^{\pm}=Y_n+a\,\Delta_n\pm b\,\Delta_n^{1/2},$$

where ΔW_n and ΔZ_n are the same as in (21). Implicit or fully implicit schemes are needed to handle stiff SDEs, which will be discussed in Section 8 of Chapter 9 and in Chapter 12; see Petersen (1987), Drummond & Mortimer (1991) and Hernandez & Spigler (1993).

Milstein, Platen & Schurz (1998) have proposed a family of balanced methods that seem to be rather effective for stiff SDEs. Implicit schemes or different concepts of numerical stability have been suggested and studied in a variety of papers, including Talay (1982b, 1984), Klauder & Petersen (1985a), Pardoux & Talay (1985), Milstein (1988a, 1995a), Smith & Gardiner (1988), McNeil & Craig (1988), Artemiev & Shkurko (1991), Drummond & Mortimer (1991), Kloeden & Platen (1995, 1992), Hernandez & Spigler (1992, 1993), Artemiev (1993a, 1993b, 1994), Saito & Mitsui (1993b), Hofmann & Platen (1994), Milstein & Platen (1994), Hofmann (1995), Komori & Mitsui (1995), Hofmann & Platen (1996), Saito & Mitsui (1996), Schurz (1996a, 1996c), Ryashko & Schurz (1997), Burrage (1998), Fischer & Platen (1998), Higham (1998), Milstein, Platen & Schurz (1998) and Petersen (1998).

Another type of strong approximations was investigated in Gorostiza (1980) and Newton (1990). In the 1-dimensional case the time is here discretized in such a way that a random walk takes place on a prescribed set of threshholds in the state space, with the approximating process remaining on a fixed level for a random duration of time and then switching with given intensity to the next level above or below it. Finally, the reader is referred to Doss (1977), Sussmann (1978) and Talay (1982b) for other investigations of strong approximations of Ito diffusions.

Weak Taylor Approximations

When we are interested only in weak approximations of an Ito process, that is a process with approximately the same probability distribution, then we have many more degrees of freedom than with strong approximations. For example, it suffices to use an initial value $Y_0 = \hat{X}_0$ with a convenient probability distribution which approximates that of X_0 in an appropriate way. In addition the random increments ΔW_n of the Wiener process can be replaced by other more convenient approximations $\Delta \hat{W}_n$ which have similar moment properties to the ΔW_n. In a weak approximation of order $\beta = 1.0$ we could, for instance, choose independent $\Delta\hat{W}_n$ for $n = 0, 1, \ldots, N-1$ with moments

$$E\left(\left(\Delta\hat{W}_n\right)^r\right) = \begin{cases} 0 & : \; r = 1 \text{ and } 3 \\ \Delta_n & : \; r = 2 \\ Z_r(\Delta_n) & : \; r = 4, 5, \ldots \end{cases} \tag{25}$$

where

$$|Z_r(\Delta_n)| \leq K\,\Delta_n^2 \tag{26}$$

for $r = 4, 5, \ldots$ and some finite constant $K > 0$. This means we could use an easily generated two-point distributed random variable taking values $\pm\sqrt{\Delta_n}$ with equal probabilities, that is with

$$P\left(\Delta\hat{W}_n = \pm\sqrt{\Delta_n}\right) = \frac{1}{2}. \tag{27}$$

The simplest useful weak Taylor approximation is the *weak Euler scheme*

$$Y_{n+1} = Y_n + a\,\Delta_n + b\,\Delta\hat{W}_n \tag{28}$$

for $n = 0, 1, \ldots, N-1$. It follows from results in Talay (1984) that (28) has weak order $\beta = 1.0$ if the coefficients a and b are four times continuously differentiable with these derivatives satisfying a growth bound. This contrasts with the order $\gamma = 0.5$ of the strong Euler scheme (15).

We can construct weak Taylor approximations of higher order $\beta = 2.0, 3.0, \ldots$ by truncating the Wagner-Platen expansion. For example, the *weak Taylor approximation of order* $\beta = 2.0$ has, following Milstein (1978) and Talay (1984), the form

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta_n + b\,\Delta\hat{W}_n + \frac{1}{2}bb'\left\{\left(\Delta\hat{W}_n\right)^2 - \Delta_n\right\} \\ &\quad + ba'\,\Delta\hat{Z}_n + \frac{1}{2}\left\{aa' + \frac{1}{2}b^2a'\right\}\Delta_n^2 \\ &\quad + \left\{ab' + \frac{1}{2}b^2b''\right\}\left\{\Delta\hat{W}_n\,\Delta_n - \Delta\hat{Z}_n\right\} \end{aligned} \tag{29}$$

for $n = 0, 1, \ldots, N-1$. Here $\Delta\hat{W}_n$ approximates ΔW_n and $\Delta\hat{Z}_n$ the multiple stochastic integral (20). As with the weak Euler scheme (28) we can choose random variables $\Delta\hat{W}_n$ and $\Delta\hat{Z}_n$ which have approximately the same moment properties as ΔW_n and ΔZ_n, For example, we could use

$$\Delta\hat{W}_n = \Delta W_n \qquad \text{and} \qquad \Delta\hat{Z}_n = \frac{1}{2}\,\Delta W_n\,\Delta_n \tag{30}$$

with the ΔW_n independent $N(0;\Delta_n)$ normally distributed, or we could use

$$\Delta\hat{W}_n = \Delta_n^{1/2}T_n \qquad \text{and} \qquad \Delta\hat{Z}_n = \frac{1}{2}\,\Delta_n^{3/2}T_n, \tag{31}$$

where the T_n are independent three-point distributed random variables with

$$(32) \qquad P\left(T_n = \pm\sqrt{3}\right) = \frac{1}{6} \quad \text{and} \quad P\left(T_n = 0\right) = \frac{2}{3}.$$

Multi-dimensional and higher order weak Taylor approximations also involve additional random variables, but these are much simpler than those in strong approximations as will be seen in Chapter 14.
It was shown under appropriate assumptions in Platen (1984) that a Taylor approximation converges with any desired weak order $\beta = 1.0, 2.0, \ldots$ when the multiple stochastic integrals up to multiplicity β are included in the truncated stochastic Taylor expansion used to construct the scheme.

Weak Runge-Kutta and Extrapolation Approximations

It is often convenient computationally to have weak approximations of Runge-Kutta type which avoid the use of derivatives of the drift and diffusion coefficients, particularly the higher order derivatives. An *order* 2.0 *weak Runge-Kutta scheme* proposed by Talay (1984) is of the form

$$\begin{aligned}(33) \quad Y_{n+1} &= Y_n + \left\{a\left(\hat{\Upsilon}_n\right) - \frac{1}{2} b\left(\hat{\Upsilon}_n\right) b'\left(\hat{\Upsilon}_n\right)\right\} \Delta_n \\ &\quad + \left\{\frac{1}{\sqrt{2}} b\left(A_n - B_n\right) + \sqrt{2} b\left(\hat{\Upsilon}_n\right) B_n\right\} \Delta_n^{1/2} \\ &\quad + \left\{\frac{1}{2}\left(b\left(\hat{\Upsilon}_n\right) b'\left(\hat{\Upsilon}_n\right) - bb'\right) B_n^2 - bb' A_n B_n\right\} \Delta_n\end{aligned}$$

with supporting value

$$\hat{\Upsilon}_n = Y_n + \frac{1}{2}\left(a - \frac{1}{2} bb'\right) \Delta_n + \frac{1}{\sqrt{2}} b A_n \Delta^{1/2} + \frac{1}{4} bb' A_n^2 \Delta_n$$

for $n = 0, 1, \ldots, N-1$, where the A_n and B_n are independent random variables which are, for example, standard normally distributed or as in (32).
The scheme (33) still uses the derivative b' of the diffusion coefficient b. It is possible to avoid such derivative, as in the following *order* 2.0 *weak Runge-Kutta scheme* due to Platen:

$$\begin{aligned}(34) Y_{n+1} &= Y_n + \frac{1}{2}\left\{a\left(\hat{\Upsilon}_n\right) + a\right\} \Delta_n + \frac{1}{4}\left\{b\left(\Upsilon_n^+\right) + b\left(\Upsilon_n^-\right) + 2b\right\} \Delta \hat{W}_n \\ &\quad + \frac{1}{4}\left\{b\left(\Upsilon_n^+\right) - b\left(\Upsilon_n^-\right)\right\}\left\{\left(\Delta \hat{W}_n\right)^2 - \Delta_n\right\} \Delta_n^{-1/2}\end{aligned}$$

with supporting values

$$\hat{\Upsilon}_n = Y_n + a\,\Delta_n + b\,\Delta\hat{W}_n \quad \text{and} \quad \Upsilon_n^{\pm} = Y_n + a\,\Delta_n \pm b\,\Delta_n^{1/2}$$

for $n = 0, 1, \ldots, N-1$, where the $\Delta\hat{W}_n$ can be chosen as in (30) or (31). Weak second and third order Runge-Kutta type schemes have been proposed, for instance, by Kloeden & Platen (1992), Mackevicius (1994) and Komori & Mitsui (1995).

Higher order approximations of functionals can also be obtained with lower order weak schemes by extrapolation methods. Talay & Tubaro (1990) proposed an *order* 2.0 *weak extrapolation method*

$$V^{\delta}_{g,2}(T) = 2E\left(g\left(Y^{\delta}(T)\right)\right) - E\left(g\left(Y^{2\delta}(T)\right)\right), \tag{35}$$

where $Y^{\delta}(T)$ and $Y^{2\delta}(T)$ are the Euler approximations at time T for the step sizes δ and 2δ, respectively. Higher order extrapolation methods from Kloeden & Platen (1991b) will also be presented in Section 3 of Chapter 15. Essentially, many order β weak schemes can be extrapolated with formulae similar to (35) to provide order 2β accuracy for $\beta = 1.0, 2.0, \ldots$. Further results on extrapolation methods can be found in Hofmann (1994), Goodlett & Allen (1994) and Mackevicius (1996). Artemiev (1985), Müller-Gronbach (1996), Gaines & Lyons (1997), Mauthner (1998) and Burrage (1998) have derived results on step size control. Furthermore, Hofmann (1994), Hofmann, Müller-Gronbach & Ritter (1998) have considered extrapolation methods with both step size and order control.

An *order* 2.0 *weak predictor-corrector scheme* for SDEs proposed by Platen, which has the corrector

$$Y_{n+1} = Y_n + \frac{1}{2}\left\{a\left(\hat{\Upsilon}_{n+1}\right) + a\right\}\Delta_n + \Psi_n \tag{36}$$

with

$$\Psi_n = b\,\Delta\hat{W}_n + \frac{1}{2}\,bb'\left\{\left(\Delta\hat{W}_n\right)^2 - \Delta_n\right\} + \frac{1}{2}\left\{ab' + \frac{1}{2}\,b^2b''\right\}\Delta\hat{W}_n\,\Delta_n$$

and the predictor

$$\hat{\Upsilon}_{n+1} = Y_n + a\,\Delta_n + \Psi_n + \frac{1}{2}\,a'b\,\Delta\hat{W}_n\,\Delta_n + \frac{1}{2}\left\{aa' + \frac{1}{2}\,b^2a''\right\}\Delta_n^2, \tag{37}$$

where $\Delta\hat{W}_n$ and Δ_n can be as in (30) or (31)–(32), is an example of a simplified weak Taylor scheme. The corrector (36) resembles the *implicit order* 2.0 *weak scheme*

$$Y_{n+1} = Y_n + \frac{1}{2}\left\{a\left(Y_{n+1}\right) + a\right\}\Delta_n + \Psi_n. \tag{38}$$

Higher order Runge-Kutta, predictor-corrector and implicit weak schemes as well as extrapolation methods will be examined in Chapter 15.

Runge-Kutta schemes with convergence only in the first two moments have been considered in Greenside & Helfand (1981), Haworth & Pope (1986), Helfand (1979), Klauder & Petersen (1985a) and Petersen (1987). This convergence criterion is weaker than the weak convergence criterion (10) considered here. Obviously, a scheme which converges with some weak order β will not only converge in the first two moments, but also in all higher moments with this same order β when they exist. Further papers that deal with weak higher order approximations and their numerical stability include Fahrmeir (1974), Milstein (1978, 1985, 1988a), Platen (1980b, 1984, 1995), Gladyshev & Milstein (1984), Talay (1984, 1986, 1990), Ventzel, Gladyshev & Milstein (1985), Haworth & Pope (1986), Talay & Tubaro (1990), Drummond & Mortimer (1991), Kloeden & Platen (1991b), Mikulevicius & Platen (1991), Kloeden, Platen & Hofmann (1992), Kannan & Wu (1993), Hofmann (1994, 1995), Hofmann & Platen (1994, 1996), Mackevicius (1994), Komori & Mitsui (1995), Bally & Talay (1996a, 1996b), Kohatsu-Higa & Ogawa (1997) and Milstein & Tretjakov (1997).

Monte-Carlo Simulation and Variance Reduction

Wagner (1987a, 1987b) has proposed another way of approximating weak approximations of diffusion processes which is based on the Monte Carlo simulation of functional integrals and uses unbiased, variance reduced approximations to estimate functionals of Ito diffusion processes. This will be described in Chapter 16. Useful references on variance reduction techniques in a more classical setting include Hammersley & Handscomb (1964), Ermakov (1975), Boyle (1977), Maltz & Hitzl (1979), Rubinstein (1981), Ermakov & Mikhailov (1982), Ripley (1983b), Kalos & Whitlock (1986), Bratley, Fox & Schrage (1987), Chang (1987), Wagner (1987a, 1988a, 1988b, 1989a, 1989b), Law & Kelton (1991) and Ross (1990). Other variance reduction methods use more structure of the underlying SDE, see e.g., Milstein (1988a), Glynn & Iglehart (1989), Goodlett & Allen (1994) and Newton (1994, 1997).

To compute functionals of diffusions also quasi Monte Carlo methods have been employed, where the random variables are replaced by elements from some low discrepancy sequence or point set, see, e.g., the book by Niederreiter (1992). Low discrepancy point sets such as Sobol, Halton or Faure sequences, discussed for instance in Halton (1960), Sobol (1967), Tezuka (1993), Tezuka & Tokuyama (1994), Radovic, Sobol & Tichy (1996), Tuffin (1996, 1997), and Mori (1998), exhibit fewer deviations from uniformity compared to uniformly distributed random point sets. This can generally lead to faster rates of convergence compared to random sequences as discussed in Hofmann & Mathé (1997) and Sloan & Wozniakowski (1998). However the gain in efficiency is not always balanced with the bias that may result from the use of these methods.

We conclude this brief survey with the remark that the theoretical understanding and practical application of numerical methods for stochastic differential equations are still under development. An aim of this book is to stimulate an interest and further work on such methods. For this the Bibliographical Notes at the end of the book may be also of assistance.